Lecture 8 :

Limit Behaviour

Last Time

Q1. Existence of stationary distributions

Q2. Uniqueness of stationary distributions

Q3. Ergodicity : Suppose there exists a unique stationary distribution $\vec{\pi}$, given $X_0 \sim \vec{\mu}$, will it always true that $\vec{\mu} \cdot P^n \xrightarrow{n \to \infty} \vec{\pi}$?

Today

1°. Existence of stationary distributions

Theorem 8.1. Suppose that the state space $\mathcal{X}$ is irreducible and that all states in $\mathcal{X}$ are recurrent. Then there is a stationary measure $\vec{\mu}$ with

$0 < \vec{\mu}_y < \infty$, for all $y \in \mathcal{X}$.

Pf. Fix $x \in \mathcal{X}$. Let $\tau_x = \min\{n \geq 1 \mid X_n = x\}$.

Claim 1:
$$\vec{\mu}_y^x := E_x[\#\text{ of visits to } y \text{ before returning to } x] = \sum_{n=0}^{\infty} \mathbb{P}_x(X_n = y, \tau_x > n)$$

is stationary, i.e. $\vec{\mu}^x \cdot P = \vec{\mu}^x$.

Prepared by Kelvin Shuangjian ZHANG (ks3zhang@waterloo.ca)

Pf of Claim 1: Notice that

$$\#\text{ of visits to } y \text{ before returning to } x = \sum_{n=0}^{\infty} \mathbb{1}\{X_n = y,\ \tau_x > n\}$$

Thus,

$$\begin{aligned}
\vec{\mu}^x_y &:= \mathbb{E}_x[\#\text{ of visits to } y \text{ before returning to } x] \\
&= \mathbb{E}_x\Big[\sum_{n=0}^{\infty} \mathbb{1}\{X_n = y,\ \tau_x > n\}\Big] \\
&= \sum_{n=0}^{\infty} \mathbb{E}_x[\mathbb{1}\{X_n = y,\ \tau_x > n\}] \\
&= \sum_{n=0}^{\infty} \mathbb{P}_x(X_n = y,\ \tau_x > n).
\end{aligned}$$

To see $\vec{\mu}^x$ is stationary,

$$\begin{aligned}
[\vec{\mu}^x P]_z &= \sum_{y \in \mathcal{X}} \vec{\mu}^x_y \cdot P_{yz} \\
&= \sum_{y \in \mathcal{X}} \sum_{n=0}^{\infty} \mathbb{P}_x(X_n = y,\ \tau_x > n) \cdot P_{yz}. \\
&= \sum_{n=0}^{\infty} \sum_{y \in \mathcal{X}} \mathbb{P}_x(X_n = y,\ \tau_x > n) \cdot P_{yz} \\
&= \sum_{n=0}^{\infty} \sum_{y \in \mathcal{X}} \mathbb{P}_x(X_n = y,\ \tau_x > n,\ X_{n+1} = z) \\
&= \sum_{n=0}^{\infty} \mathbb{P}_x(X_{n+1} = z,\ \tau_x > n).
\end{aligned}$$

Case I: $z \neq x$.

$$[\vec{\mu}^x P]_z = \sum_{n=0}^{\infty} \mathbb{P}_x(X_{n+1} = z, \tau_x > n)$$

$$= \sum_{n=0}^{\infty} \mathbb{P}_x(X_{n+1} = z, \tau_x > n+1)$$

$$= \sum_{n=1}^{\infty} \mathbb{P}_x(X_n = z, \tau_x > n)$$

$$= \sum_{n=0}^{\infty} \mathbb{P}_x(X_n = z, \tau_x > n)$$

$$= \vec{\mu}^x_z.$$

Case II: $z = x$

$$[\vec{\mu}^x P]_x = \sum_{n=0}^{\infty} \mathbb{P}_x(X_{n+1} = z, \tau_x > n)$$

$$= \sum_{n=0}^{\infty} \mathbb{P}_x(\tau_x = n+1)$$

$$= \mathbb{P}_x(\tau_x < \infty)$$

x is recurrent

$$= P_{xx} = 1$$

On the other hand, $\vec{\mu}^x_x = \sum_{n=0}^{\infty} \mathbb{P}_x(X_n = x, \tau_x > n) = 1.$

Thus, $[\vec{\mu}^x P]_x = \vec{\mu}^x_x$.

Therefore, $\vec{\mu}^x P = \vec{\mu}^x$, i.e. $\vec{\mu}^x$ is stationary. $\square$

Claim 2: $\forall y \in \mathcal{X}$, $\vec{\mu}^x_y < \infty$.

Pf of Claim 2: Since $\mathcal{X}$ is irreducible, for any $y \in \mathcal{X}$, $y \longrightarrow x$. Thus, there exists $n \geq 1$, such that $[P^n]_{yx} > 0$. Notice

$$\vec{\mu}^x_y \cdot [P^n]_{yx} \leq \sum_{z \in \mathcal{X}} \vec{\mu}^x_z \cdot [P^n]_{zx} = [\vec{\mu}^x P^n]_x = \vec{\mu}^x_x = 1.$$

Thus, $\vec{\mu}^x_y \leq \frac{1}{[P^n]_{yx}} < \infty$. □

Claim 3: $\forall y \in \mathcal{X}$, $\vec{\mu}^x_y > 0$.

Pf of Claim 3: Case I: If $y = x$, then $\vec{\mu}^x_x = 1 > 0$.

Case II: If $y \neq x$. Since $\mathcal{X}$ is irreducible, $x \longrightarrow y$.

Define $K := \min\{k : [P^k]_{xy} > 0\}$, then K is finite.

This implies $\exists y_1, y_2, \cdots, y_{K-1} \in \mathcal{X}$ s.t. $P_{xy_1} \cdot P_{y_1y_2} \cdot \cdots \cdot P_{y_{K-1}y} > 0$; and by minimality, $y_i \neq x$, $\forall i \in [K-1]$.

Therefore, $\vec{\mu}^x_y \geq \mathbb{P}_x(X_K = y, \tau_x > K) \geq P_{xy_1} \cdot P_{y_1y_2} \cdot \cdots \cdot P_{y_{K-1}y} > 0$. □

Thus, this $\vec{\mu}^x$ is a stationary measure with $\vec{\mu}^x_y \in (0, \infty)$, $\forall y \in \mathcal{X}$. ■

Remark 8.1. Theorem 8.1 describes the existence of stationary measures. If, in addition, $\mathcal{X}$ is finite, then $\sum_{y\in\mathcal{X}} \vec{\mu}^{x}_{y} < \infty$ and $\vec{\pi} = \dfrac{\vec{\mu}^{x}}{\sum_{y\in\mathcal{X}} \vec{\mu}^{x}_{y}}$ is a stationary distribution. For cases of infinite state spaces, being irreducible and recurrent (i.e. all the states are recurrent) is not sufficient. We provide an example below now and will introduce another existence theorem for the cases of infinite state spaces.

Ex 8.1. (Reflecting SRW on $\mathbb{Z}$). Let $\mathcal{X} = \mathbb{N}$. The transition probability

$$\mathbb{P}(X_1 = y \mid X_0 = x) = \begin{cases} \frac{1}{2}, & y = x+1; \\ \frac{1}{2}, & y = x-1 \ (x>0); \\ \frac{1}{2}, & y = x = 0; \\ 0, & \text{else.} \end{cases}$$

why? In this case, the chain is irreducible and recurrent,

why? but there is no stationary distribution.

2°. We will postpone the theory of uniqueness.

Ergodicity: Given $X_0 \sim \vec{\mu}$, will $\vec{\mu} P^n \xrightarrow{n\to\infty} \vec{\pi}$?

Suppose $\vec{\mu} = \delta_x$, then $[\vec{\mu} P^n]_y = [P^n]_{xy} \xrightarrow{n\to\infty} \vec{\pi}_y$.

Thus, $[P^n]_{xy} \xrightarrow{n\to\infty} \vec{\pi}_y$, $\forall x \in \mathcal{X}$.

On the other hand, suppose $[P^n]_{xy} \xrightarrow{n\to\infty} \vec{\pi}_y$, $\forall x \in \mathcal{X}$,

then
$$\lim_{n\to\infty} [\vec{\mu} P^n]_y = \lim_{n\to\infty} \sum_{x\in\mathcal{X}} \vec{\mu}_x \cdot [P^n]_{xy}$$

Tannery's theorem

$$= \sum_{x\in\mathcal{X}} \vec{\mu}_x \lim_{n\to\infty} [P^n]_{xy}$$
$$= \sum_{x\in\mathcal{X}} \vec{\mu}_x \vec{\pi}_y$$
$$= \vec{\pi}_y \cdot \sum_{x\in\mathcal{X}} \vec{\mu}_x = \vec{\pi}_y \cdot 1 = \vec{\pi}_y .$$

Thus $\boxed{\vec{\mu} P^n \xrightarrow{n\to\infty} \vec{\pi} \text{ iff } [P^n]_{xy} \xrightarrow{n\to\infty} \vec{\pi}_y, \ \forall x \in \mathcal{X}.}$

Remark 8.2 Suppose y is transient. then $\forall x \in \mathcal{X}$,

$$\sum_{n=1}^{\infty} [P^n]_{xy} = \mathbb{E}_x N(y) = \frac{\rho_{xy}}{1-\rho_{yy}} < \infty .$$

Thus, $[P^n]_{xy} \xrightarrow{n\to\infty} 0$, $\forall x \in \mathcal{X}$.

$\pi = \pi P^n$, $\forall n \in \mathbb{N}$

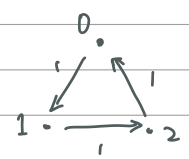
Since $\vec{\pi}_y = [\vec{\pi} P^n]_y = \sum_{x\in\mathcal{X}} \vec{\pi}_x [P^n]_{xy}$, $\forall n \in \mathbb{N}$,

$$\vec{\pi}_y = \lim_{n\to\infty} \sum_{x\in\mathcal{X}} \vec{\pi}_x [P^n]_{xy} = \sum_{x\in\mathcal{X}} \vec{\pi}_x \lim_{n\to\infty} [P^n]_{xy} = 0.$$

Therefore, $[P^n]_{xy} \xrightarrow{n\to\infty} \vec{\pi}_y$, $\forall x \in \mathcal{X}$.

By the Decomposition Theorem, we can focus on an irreducible recurrent set.

Ex 8.2



$$[P^n]_{00} = \begin{cases} 1, & n \equiv 0 \pmod 3 \\ 0, & \text{else}. \end{cases}$$

Def 8.1 (Periodicity). A state x has period d_x if

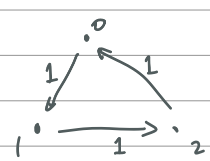
$d_x = \gcd(\{n \geq 1 \mid [P^n]_{xx} > 0\})$.

A state x is called aperiodic if $d_x = 1$.

A state x is called periodic if $d_x > 1$.

Denote by $I_x = \{n \geq 1 \mid [P^n]_{xx} > 0\}$, then $d_x = \gcd(I_x)$.

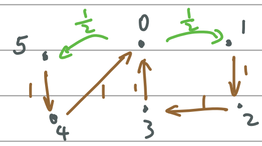
Ex 8.3.



$I_0 = \{3, 6, 9, \cdots\}$

$\Rightarrow d_0 = 3$.

Ex 8.4



$I_0 = \{3,4\} \cup \{3a+4b \mid a,b \geq 0, a+b \geq 1\}$

$d_0 = \gcd(I_0) = 1.$

$I_1 = \{4\} \cup \{4a+3b \mid a \geq 1, b \geq 0\} \Rightarrow d_1 = \gcd(I_1) = 1.$

$d_0 = d_1$!

Lemma 8.1. If $x \to y$ and $y \to x$, then $d_x = d_y$.

Pf. (Proof by contradiction).

Without loss of generality (WLOG), suppose $c = d_x > d_y$.

Since $x \longrightarrow y$ and $y \longrightarrow x$, there exist $k, m \geq 1$, s.t. $[P^k]_{xy} > 0$, $[P^m]_{yx} > 0$.

This implies, $[P^{k+m}]_{xx} \geq [P^k]_{xy} \cdot [P^m]_{yx} > 0$.

Thus, $k+m \in I_x$. Therefore, $k+m \equiv 0 \pmod{c}$.

For any $l \in I_y$, one has $[P^l]_{yy} > 0$ and thus

$$[P^{k+l+m}]_{xx} \geq [P^k]_{xy} [P^l]_{yy} [P^m]_{yx} > 0.$$

This implies, $k+l+m \in I_x$, i.e. $k+l+m \equiv 0 \pmod{c}$.

Therefore, $\ell \equiv 0 \pmod{c}$, $\forall \ell \in I_y$.

That is, $c = d_x \mid \ell$, $\forall \ell \in I_y$.

Thus, $d_x \mid d_y$. This implies $d_x \le d_y < d_x$.

This is a contradiction. ■

Def 8.2. A property K of a state is called a class property if, whenever $x \longrightarrow y$ & $y \longrightarrow x$,

x has $K \iff y$ has K.

Remark 8.3. Periodicity is a class property.

Q: What are other class properties?

Lemma 8.2. I_x is closed under addition. That is,

$a, b \in I_x \implies a+b \in I_x$.

Lemma 8.3. If x is aperiodic, then there is an n_0 such that all $n \ge n_0$ are in I_x.

Lemma 8.4. $P_{xx} > 0 \implies x$ is aperiodic.

This is the end of this lecture!